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COMPARISON OF SOME STRESS RATES

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Abstract—This paper introduces a new derivation rule by means of which two new stress rates are defined. Furthermore, the paper compares five stress rates: Jaumann's, Truesdell's, Green–Naghdi's, Sowerby–Chu's and Durban–Baruch's, for simple shear. Elastic, elastic–perfectly plastic and elastic–plastic hardening (isotropic, kinematic, and combined) material models are considered. Different solutions have already been published for these cases, except for Sowerby–Chu and Durban–Baruch time derivatives. Using the new derivation rule the new rate form of the hyperelastic Doyle–Ericksen formula is obtained. Taking advantage of the Sowerby–Chu stress rate a new constitutive equation for hypoelastic material is given.

INTRODUCTION

One important field of research of constitutive equations is the investigation of stress rates. Different stress rates have been analysed by many authors. Most of the investigations included homogeneous large deformation problems like pure tension, simple shear (Moss, 1984; Sowerby and Chu, 1984; Johnson and Bammann, 1984; Atluri, 1984b; Reed and Atluri, 1983; Dienes, 1979; Dafalias, 1983; Lee *et al.*, 1983; Nagtegaal and de Jong, 1982; Key, 1984, using classical stress rates (Jaumann, Truesdell, Oldroyd, Cotter–Rivlin, Green–Naghdi) and their modified versions for the solutions to constitutive equations. The solutions were mostly obtained for simple hypoelastic (Sowerby and Chu, 1984; Atluri, 1984b; Reed and Atluri, 1983; Dienes, 1979; Truesdell, 1955; Key, 1984; Reed and Atluri, 1985, rigid–plastic isotropic (Lee *et al.*, 1983; Nagtegaal and de Jong, 1982), and kinematic hardening (Dafalias, 1983; Lee *et al.*, 1983; Nagtegaal and de Jong, 1982; Reed and Atluri, 1985; Paulun and Pecherski, 1985), as well as elastic–plastic isotropic and kinematic hardening (Johnson and Bammann, 1984; Atluri, 1984b; Key, 1984) materials. The elastic–ideally plastic isotropic material was investigated by Moss (1984) using the Prandtl–Reuss flow rule. Conclusions drawn from the results are detailed in the works of Key (1984), Atluri (1984b), Nagtegaal and de Jong (1982), Lee *et al* (1983), and Dafalias (1983).

The same solutions can be obtained also for the Sowerby–Chu (1984) and Durban–Baruch (1977) stress rates that have been published in recent years. For analysis of the results so obtained, these solutions will be compared with existing solutions for stress rates.

The present work contains this comparison for the case of simple shear using elastic, elastic–perfectly plastic, elastic–plastic isotropic, kinematic, and combined isotropic–kinematic hardening models.

In the first part of this paper, the stress rates and their derivation are reviewed. A unified formulation of the stress rates and a new stress rate definition are given.

The second part contains solutions for simple shear. For the elastic case, the solutions can be obtained analytically while in the elastic–perfectly plastic case, the relationships lead to one- or two-dimensional differential equations. For the elastic–plastic hardening material, the isotropic, kinematic, and combined model can be described by means of a nine-dimensional differential equation system. Kinematic hardening is taken into consideration by use of the Prager model. No approximations are contained in the relationships derived. The system of differential equations is solved numerically using the fourth-order Runge–Kutta method. Some conclusions can be drawn from the evaluation of the obtained results that may contribute to the synthesis of stress rates.

Finally, the constitutive equations are analysed. The new derivation rule is used to develop a rate form of the Doyle–Ericksen formula and a new constitutive equation for hypoelastic materials.

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STRESS RATES

The most important stress rates fulfilling the physical objectivity for Cauchy's stress tensor $\mathbf{t}$ are given in Table 1. Notations used in Table 1 are as follows: $(\dot{\ })$, material time derivative, $\mathbf{L}$, velocity gradient tensor: $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, where $\mathbf{F}$ is the deformation gradient tensor, $\mathbf{D}$ the strain rate of deformation: $\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^T)$, $\mathbf{W}$ the spin tensor: $\mathbf{W} = \frac{1}{2}(\mathbf{L}-\mathbf{L}^T)$, $\boldsymbol{\Omega}$ the rate of the rotation tensor: $\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$, where $\mathbf{R}$ is the orthogonal tensor in polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{1}$$

of deformation gradient $\mathbf{F}$. $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, $\boldsymbol{\Omega}_E$ the spin tensor: $\boldsymbol{\Omega}_E = \dot{\mathbf{R}}_E\mathbf{R}_E^T$, where $\mathbf{R}_E$ is the diagonal transformation of stretch tensor $\mathbf{V}$

$$\mathbf{V} = \mathbf{R}_E\boldsymbol{\lambda}\mathbf{R}_E^T \tag{2}$$

and $\boldsymbol{\lambda}$ the diagonal tensor containing the eigenvalues of $\mathbf{V}$.

For the stress rates listed in Table 1 the following definitions are known. The Truesdell (1955) time derivative, using the Kirchhoff stress tensor

$$\mathring{\mathbf{t}} = \frac{1}{J}\mathbf{F}(\dot{\overline{\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}}})\mathbf{F}^T \tag{3}$$

where $\boldsymbol{\tau} = J\mathbf{t}$, $J = \det(\mathbf{F})$, or briefly, with the second Piola–Kirchhoff stress tensor

$$\mathring{\mathbf{t}} = \frac{1}{J}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^T. \tag{4}$$

The Jaumann (1911) rate can be produced as the average of the Oldroyd (1950) rate

$$\breve{\mathbf{t}} = \dot{\mathbf{t}} - \mathbf{L}\mathbf{t} - \mathbf{t}\mathbf{L}^T \tag{5}$$

and the Cotter–Rivlin (1955), or convected stress rate

$$\hat{\mathbf{t}} = \dot{\mathbf{t}} + \mathbf{L}^T\mathbf{t} + \mathbf{t}\mathbf{L} \tag{6}$$

as

$$\overset{\vee}{\mathbf{t}} = \tfrac{1}{2}(\breve{\mathbf{t}} + \hat{\mathbf{t}}). \tag{7}$$

The Green–Naghdi (1965), or Green–McInnis (1967), or Dienes (1979), stress rate can be written as

$$\overset{\square}{\mathbf{t}} = \mathbf{R}(\dot{\overline{\mathbf{R}^T\mathbf{t}\mathbf{R}}})\mathbf{R}^T \tag{8}$$

Table 1

Truesdell	$\mathring{\mathbf{t}} = \dot{\mathbf{t}} - \mathbf{L}\mathbf{t} - \mathbf{t}\mathbf{L}^T + \mathbf{t}\,\mathrm{tr}\,(\mathbf{D})$
Jaumann	$\overset{\vee}{\mathbf{t}} = \dot{\mathbf{t}} - \mathbf{W}\mathbf{t} + \mathbf{t}\mathbf{W}$
Green–Naghdi, Green–McInnis, Dienes	$\overset{\square}{\mathbf{t}} = \dot{\mathbf{t}} - \boldsymbol{\Omega}\mathbf{t} + \mathbf{t}\boldsymbol{\Omega}$
Sowerby–Chu	$\overset{\bullet}{\mathbf{t}} = \dot{\mathbf{t}} - \boldsymbol{\Omega}_E\mathbf{t} + \mathbf{t}\boldsymbol{\Omega}_E$
Durban–Baruch	$\overset{N}{\mathbf{t}} = \dot{\mathbf{t}} - (\frac{1}{2}\mathbf{D} + \mathbf{W})\mathbf{t} + \mathbf{t}(\mathbf{W} - \frac{1}{2}\mathbf{D}) + \mathbf{t}\,\mathrm{tr}\,(\mathbf{D})$

or briefly, with Cauchy's rotated tensor $\mathbf{T} = \mathbf{R}^T\mathbf{t}\mathbf{R}$

$$\overset{\square}{\mathbf{t}} = \mathbf{R}\dot{\mathbf{T}}\mathbf{R}^T. \tag{9}$$

The definition of the Sowerby–Chu (1984) time derivative is

$$\overset{\bullet}{\mathbf{t}} = \mathbf{R}_E\overline{(\mathbf{R}_E^T\mathbf{t}\mathbf{R}_E)}^{\bullet}\mathbf{R}_E^T. \tag{10}$$

The Durban–Baruch (1977) (natural) stress rate can be produced as the average of the Truesdell rate, and of the Jaumann rate of the Kirchhoff stress tensor

$$\overset{N}{\mathbf{t}} = \frac{1}{2}\left(\overset{\circ}{\mathbf{t}} + \frac{1}{J}\overset{\nabla}{\boldsymbol{\tau}}\right). \tag{11}$$

Remark 1. A rather elegant way to derive stress rates is given by Marsden and Hughes (1983) and Simo and Marsden (1984). Using the Lie derivation, the stress rates described above can be produced. The Lie derivation on the covariant coordinates of the stress tensor results in the Cotter–Rivlin rate. The Oldroyd rate can be obtained using the Lie derivation on the contravariant stress coordinates.

A connection can be established between the Sowerby–Chu and the Green–Naghdi stress rates. A relation between $\boldsymbol{\Omega}_E$ and $\boldsymbol{\Omega}$ is given by (Chu, 1986)

$$\boldsymbol{\Omega}_E = \boldsymbol{\Omega} + \mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T \tag{12}$$

where

$$\boldsymbol{\Omega}_L = \dot{\mathbf{R}}_L\mathbf{R}_L^T$$

and $\mathbf{R}_L$ is a diagonal transformation of tensor $\mathbf{U}$

$$\mathbf{U} = \mathbf{R}_L\boldsymbol{\lambda}\mathbf{R}_L^T. \tag{13}$$

Using eqn (12)

$$\overset{\bullet}{\mathbf{t}} = \overset{\square}{\mathbf{t}} - \mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T\mathbf{t} + \mathbf{t}\mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T. \tag{14}$$

The Sowerby–Chu derivative of Cauchy's rotated stress tensor $\mathbf{T}$, in terms of tensor $\boldsymbol{\Omega}_L$, takes the following shape:

$$\overset{\bullet}{\mathbf{T}} = \dot{\mathbf{T}} - \boldsymbol{\Omega}_L\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}_L. \tag{15}$$

Using eqn (15)

$$\overset{\bullet}{\mathbf{t}} = \mathbf{R}\overset{\bullet}{\mathbf{T}}\mathbf{R}^T. \tag{16}$$

Equation (16) is equivalent to eqn (14), because

$$\overset{\bullet}{\mathbf{t}} = \mathbf{R}\overset{\bullet}{\mathbf{T}}\mathbf{R}^T = \mathbf{R}(\dot{\mathbf{T}} - \boldsymbol{\Omega}_L\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}_L)\mathbf{R}^T = \mathbf{R}\dot{\mathbf{T}}\mathbf{R}^T - \mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T\mathbf{R}\mathbf{T}\mathbf{R}^T + \mathbf{R}\mathbf{T}\mathbf{R}^T\mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T$$
$$= \overset{\square}{\mathbf{t}} - \mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T\mathbf{t} + \mathbf{t}\mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T.$$

The definitions of stress rates can be illustrated by means of Fig. 1 showing configurations C_R and C_V resulting from polar decomposition of deformation gradient (1) as well as configurations $C_{\lambda 0}$ and $C_{\lambda t}$ associated with the diagonal transformation of tensors U and V. For example the Truesdell, Oldroyd, and Cotter–Rivlin rates can be produced by

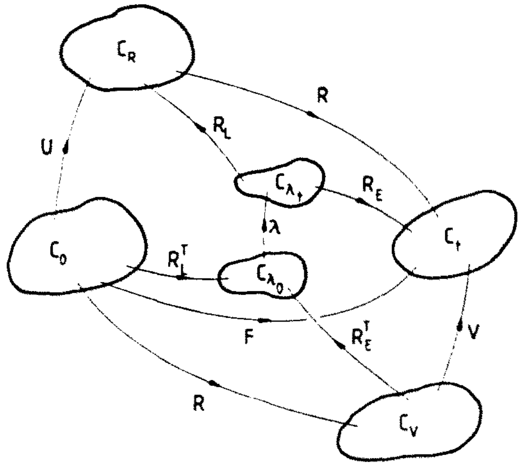

Fig. 1. Schematic representation of configurations.

the following steps: (i) a transformation of (instantaneous) configuration C_t into the (initial) configuration C_0; (ii) differentiation with respect to time; (iii) retransformation. The Green–Naghdi, or Dienes rate and the Sowerby–Chu rate can be derived in a similar way but by transformation into configuration C_R and $C_{\lambda t}$, respectively, instead of C_0. Configurations C_V and $C_{\lambda 0}$ have not yet been applied for the derivation of stress rates. It can be shown in the case of configuration C_V that the derivation procedure previously described leads to loss of physical objectivity. However, configuration $C_{\lambda 0}$ results in objective stress rates by means of the following new derivation rule.

(i) First transform a tensor (stress, strain, etc.) from the spatial configuration C_t into configuration C_V, then transform it into configuration $C_{\lambda 0}$.

(ii) Differentiate it with time.

(iii) Retransform it in the opposite direction as described in (i).

Using this derivation rule on the contravariant or covariant Cauchy stress tensor $\mathbf{t}$, the following new stress rates are obtained

$$\breve{\mathbf{t}}_E = \dot{\mathbf{t}} - \mathbf{L}_E\mathbf{t} - \mathbf{t}\mathbf{L}_E^T \tag{17}$$

$$\hat{\mathbf{t}}_E = \dot{\mathbf{t}} + \mathbf{L}_E^T\mathbf{t} + \mathbf{t}\mathbf{L}_E \tag{18}$$

where

$$\mathbf{L}_E = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}_E\mathbf{V}^{-1}. \tag{19}$$

Proposition 1. The average of stress rates $\breve{\mathbf{t}}_E$ and $\hat{\mathbf{t}}_E$ gives the Sowerby–Chu stress rate

$$\overset{\bullet}{\mathbf{t}} = \tfrac{1}{2}(\breve{\mathbf{t}}_E + \hat{\mathbf{t}}_E). \tag{20}$$

Proof. To accept this, it is enough to prove

$$\boldsymbol{\Omega}_E = \tfrac{1}{2}(\mathbf{L}_E - \mathbf{L}_E^T) = (\mathbf{L}_E)_A \tag{21}$$

where $(\ )_A$ denotes the antisymmetric part of eqn (19).

With eqn (12) substituted into eqn (19) for $\mathbf{L}_E$ we obtain

$$\mathbf{L}_E = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}\mathbf{V}^{-1} + \mathbf{F}\boldsymbol{\Omega}_L\mathbf{F}^{-1}. \tag{22}$$

The relationships

$$\mathbf{L} = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}\mathbf{V}^{-1} \tag{23}$$

and

$$\mathbf{W} = \boldsymbol{\Omega}_E - \tfrac{1}{2}(\mathbf{F}\boldsymbol{\Omega}_L\mathbf{F}^{-1} + \mathbf{F}^{-T}\boldsymbol{\Omega}_L\mathbf{F}^T) \tag{24}$$

found by Sowerby and Chu (1984) and Chu (1986) help to prove eqn (21).

Substituting eqn (23) into eqn (22) we obtain

$$\mathbf{L}_E = \mathbf{L} + \mathbf{F}\boldsymbol{\Omega}_L\mathbf{F}^{-1}. \tag{25}$$

The antisymmetric part of eqn (25) is

$$(\mathbf{L}_E)_A = (\mathbf{L})_A + (\mathbf{F}\boldsymbol{\Omega}_L\mathbf{F}^{-1})_A = \mathbf{W} + (\mathbf{F}\boldsymbol{\Omega}_L\mathbf{F}^{-1})_A. \tag{26}$$

Comparing eqn (24) with eqn (26) it follows that

$$(\mathbf{L}_E)_A = \boldsymbol{\Omega}_E. \quad \square \tag{27}$$

A lumped representation of part of the stress rates is possible using a formula introduced by Hill (1970)

$$\hat{\mathbf{t}} = \overset{\nabla}{\mathbf{t}} - m(\mathbf{D}\mathbf{t} + \mathbf{t}\mathbf{D}). \tag{28}$$

Equation (28) yields the Jaumann rate in the case of $m = 0$, the Oldroyd rate for $m = 1$, while the Cotter–Rivlin rate for $m = -1$. When the right-hand side of eqn (28) is completed with the term $\mathbf{t}\,\mathrm{tr}\,(\mathbf{D})$, the Truesdell rate is obtained for $m = 1$ while the Durban–Baruch rate is obtained for $m = 1/2$.

Another generalization of stress rates was given by Fressengeas and Molinari (1983). They used the relationship given by Mandel (1973) for the directional derivative of the deformation gradient

$$\overset{D}{\mathbf{F}} = \dot{\mathbf{F}} - \boldsymbol{\Omega}_D\mathbf{F}.$$

The stress rate was written as

$$\hat{\mathbf{t}} = \dot{\mathbf{t}} - \boldsymbol{\Omega}_D\mathbf{t} + \mathbf{t}\boldsymbol{\Omega}_D$$

where

$$\boldsymbol{\Omega}_D = \boldsymbol{\Omega} - \frac{v}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T. \tag{29}$$

Equation (29) gives the Jaumann rate for $v = 1$ and the Green–Naghdi rate for $v = 0$.

Taking the Sowerby–Chu derivative as a starting point and using eqn (12), a new generalization of the stress rates is possible:

$$\hat{\mathbf{t}} = \dot{\mathbf{t}} - (\boldsymbol{\Omega} + a\mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T)\mathbf{t} + \mathbf{t}(\boldsymbol{\Omega} + a\mathbf{R}\boldsymbol{\Omega}_L\mathbf{R}^T).$$

The above relationship yields the Green–Naghdi rate for $a = 0$ and the Sowerby–Chu rate for $a = 1$.

Investigation of the expressions for stress rates discussed in this work shows that any of them can be given in the following general form:

$$\hat{\mathbf{t}} = \dot{\mathbf{t}} - 2(\mathbf{A}\mathbf{t})_S \tag{30}$$

Table 2

Stress rates	$\mathbf{A}$
Truesdell	$\mathbf{L}-\frac{1}{2}\mathbf{I}\,\mathrm{tr}\,(\mathbf{D})$
Oldroyd	$\mathbf{L}$
Cotter–Rivlin	$-\mathbf{L}^T$
Jaumann	$\mathbf{W}$
Durban–Baruch	$\frac{1}{2}\mathbf{D}+\mathbf{W}-\frac{1}{2}\mathbf{I}\,\mathrm{tr}\,\mathbf{D}$
Green–Naghdi	$\mathbf{\Omega}$
New stress rate, eqn (17)	$\mathbf{L}_E = \dot{\mathbf{V}}\mathbf{V}^{-1}+\mathbf{V}\mathbf{\Omega}_E\mathbf{V}^{-1}$
New stress rate, eqn (18)	$-\mathbf{L}_E^T = -\mathbf{V}^{-1}\dot{\mathbf{V}}+\mathbf{V}^{-1}\mathbf{\Omega}_E\mathbf{V}$
Sowerby–Chu	$\mathbf{\Omega}_E$

where $(\ )_S$ denotes the symmetric part. Taking the objectivity condition for $\hat{\mathbf{t}}$ as a starting point, the transformation rule

$$\bar{\mathbf{A}} = \mathbf{Q}\mathbf{A}\mathbf{Q}^T+\dot{\mathbf{Q}}\mathbf{Q}^T \tag{31}$$

can be derived for quantity $\mathbf{A}$, where $\mathbf{Q}$ is an arbitrary orthogonal tensor. Hence, a physically objective stress rate can be produced with any $\mathbf{A}$ transformable according to eqn (31).

Quantities $\mathbf{A}$, associated with the stress rates investigated, are given in Table 2.

COMPARISON OF STRESS RATES IN THE CASE OF SIMPLE SHEAR

Stress rates in simple shear

For simple shear, the motion can be given in the following well-known form:

$$x_1 = X_1+e(t)X_2; \quad x_2 = X_2; \quad x_3 = X_3. \tag{32}$$

On the basis of eqns (32), the kinematic quantities required for the stress rates given in Table 1 are

$$\mathbf{L} = \dot{e}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{D} = \frac{\dot{e}}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{W} = \frac{\dot{e}}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{\Omega} = \dot{\beta}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Omega}_E = \frac{\dot{\beta}}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \beta = \tan^{-1}\left(\frac{e}{2}\right); \quad \dot{\beta} = \frac{2\dot{e}}{e^2+4}.$$

Non-zero elements of stress rates for simple shear can be summed up as

$$\hat{t}_{11} = \dot{t}_{11}+\dot{e}c_1t_{12} \tag{33}$$

$$\hat{t}_{22} = \dot{t}_{22}+\dot{e}c_2t_{12} \tag{34}$$

$$\hat{t}_{33} = \dot{t}_{33} \tag{35}$$

Table 3

Stress rates	c_1	c_2
Truesdell	-2	0
Jaumann	-1	1
Green–Naghdi	$-\frac{4}{e^2+4}$	$\frac{4}{e^2+4}$
Sowerby–Chu	$-\frac{2}{e^2+4}$	$\frac{2}{e^2+4}$
Durban–Baruch	$-\frac{3}{2}$	$\frac{1}{2}$

$$\hat{t}_{12} = \dot{t}_{12} + \tfrac{1}{2}\dot{e}(c_2 t_{11} + c_1 t_{22}). \tag{36}$$

Parameters c_1 and c_2 are included in Table 3.

Elastic solution

For the elastic solution, the simple hypoelastic relation has been used

$$\hat{\mathbf{t}} = 2\mu\mathbf{D} + \lambda\mathbf{I}\ \mathrm{tr}\ \mathbf{D} \tag{37}$$

where μ and λ are Lamé constants and $\hat{\mathbf{t}}$ is one of the stress rates.

Except for the Sowerby–Chu and the Durban–Baruch stress rates, exact solutions are given by Moss (1984), Atluri (1984b), and Dienes (1979). These solutions, along with analytical solutions for the two derivatives mentioned above, are given in Table 4.

Elastic–plastic solution

The elastic–plastic solution is based on the Prandtl–Reuss equation in the case of combined isotropic–kinematic hardening. Kinematic hardening is assumed according to the Prager model. The constitutive equation on the basis of Hughes (1984) is

$$\hat{\mathbf{t}} = 2\mu\mathbf{D} + \lambda\mathbf{I}\ \mathrm{tr}\ (\mathbf{D}) - \frac{3\mu^2}{k^2(3\mu+H')}\bar{\mathbf{s}}\bar{\mathbf{s}} : \mathbf{D} + \frac{3\mu}{2k^2(3\mu+H')}\bar{\mathbf{s}}\bar{\mathbf{s}} : [\mathbf{f}(\mathbf{t}) - \mathbf{f}(\boldsymbol{\alpha})] \tag{38}$$

$$\hat{\boldsymbol{\alpha}} = (1-\beta)\frac{\mu H'}{k^2(3\mu+H')}\bar{\mathbf{s}}\bar{\mathbf{s}} : \mathbf{D} - (1-\beta)\frac{H'}{2k^2(3\mu+H')}\bar{\mathbf{s}}\bar{\mathbf{s}} : [\mathbf{f}(\mathbf{t}) - \mathbf{f}(\boldsymbol{\alpha})] \tag{39}$$

Table 4

Truesdell	$t_{11} = \mu e^2;\quad t_{22} = t_{33} = 0$ $t_{12} = \mu e$
Jaumann	$t_{11} = -t_{22} = \mu[1-\cos(e)]$ $t_{12} = \mu\sin(e);\quad t_{33} = 0$
Green–Naghdi	$t_{11} = 4\mu[\cos(2\beta)\ln(\cos\beta) + \beta\sin(2\beta) - \sin^2\beta]$ $t_{12} = 2\mu\cos(2\beta)[2\beta - 2\tan(2\beta)\ln(\cos\beta) - \tan\beta]$ $t_{33} = 0;\quad t_{22} = -t_{11}$
Sowerby–Chu	$t_{11} = 2\mu\left[\sin(\beta)\ln\left(\frac{\sin(\beta)+1}{\cos(\beta)}\right) + \cos(\beta) - 1\right]$ $t_{12} = 2\mu\left[\cos(\beta)\ln\left(\frac{\sin(\beta)+1}{\cos(\beta)}\right) + \tan(\beta) - \sin(\beta)\right]$ $t_{33} = 0;\quad t_{22} = -t_{11}$
Durban–Baruch	$t_{11} = -3t_{22} = 2\mu\left[1-\cos\left(\frac{\sqrt{3}}{2}e\right)\right]$ $t_{12} = \frac{2}{\sqrt{3}}\mu\sin\left(\frac{\sqrt{3}}{2}e\right);\quad t_{33} = 0$

$$\dot{k} = \beta \frac{\mu H'}{k(3\mu + H')} \bar{\mathbf{s}} : \mathbf{D} - \beta \frac{H'}{2k(3\mu + H')} \bar{\mathbf{s}} : [\mathbf{f}(\mathbf{t}) - \mathbf{f}(\boldsymbol{\alpha})] \tag{40}$$

where

$$k = \sqrt{(\tfrac{1}{2}\bar{\mathbf{s}} : \bar{\mathbf{s}})}$$

$$\bar{\mathbf{s}} = \mathbf{s} - \boldsymbol{\alpha}, \quad \bar{\mathbf{t}} = \mathbf{t} - \boldsymbol{\alpha}$$

$$\mathbf{f}(\mathbf{t}) = \hat{\mathbf{t}} - \dot{\mathbf{t}}, \quad \mathbf{f}(\boldsymbol{\alpha}) = \hat{\boldsymbol{\alpha}} - \dot{\boldsymbol{\alpha}}$$

$\mathbf{s}$ is the deviatoric stress tensor, $\boldsymbol{\alpha}$ the "back-stress" tensor, β the parameter determining the proportion of isotropic and kinematic hardening, and H' the slope of the "true stress logarithmic plastic strain curve" in a uniaxial tension experiment.

Remark 2. If in eqn (30) tensor $\mathbf{A}$ is skew symmetric then the last terms vanish on the right-hand side of eqns (38)–(40).

Remark 3. When deriving eqns (38)–(40) the material derivative is used in the consistency condition of the yield surface. For other stress rates (including Oldroyd and Cotter–Rivlin derivatives) the objective stress was used in the consistency condition by Atluri (1984b). Consequently in the constitutive equations, eqns (38)–(40), the last terms are missing. Detailed analysis is given in Szabó (1988).

With constitutive equations, eqns (38)–(40), applied to simple shear and compared with eqns (33)–(36), the following differential equation system consisting of nine equations is obtained:

$$\dot{t}_{11} = \dot{e}\left[-A\bar{t}_{12}\bar{s}_{11} - c_1 t_{12} + \frac{A}{2\mu}\bar{s}_{11}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{41}$$

$$\dot{t}_{22} = \dot{e}\left[-A\bar{t}_{12}\bar{s}_{22} - c_2 t_{12} + \frac{A}{2\mu}\bar{s}_{22}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{42}$$

$$\dot{t}_{33} = \dot{e}\left[-A\bar{t}_{12}\bar{s}_{33} + \frac{A}{2\mu}\bar{s}_{33}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{43}$$

$$\dot{t}_{12} = \dot{e}\left[-A\bar{t}_{12}^2 - \tfrac{1}{2}c_2 t_{11} - \tfrac{1}{2}c_1 t_{22} + \frac{A}{2\mu}\bar{t}_{12}^2(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{44}$$

$$\dot{\alpha}_{11} = \dot{e}\left[B\bar{t}_{12}\bar{s}_{11} - c_1\alpha_{12} - \frac{B}{2\mu}\bar{s}_{11}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{45}$$

$$\dot{\alpha}_{22} = \dot{e}\left[B\bar{t}_{12}\bar{s}_{22} - c_2\alpha_{12} - \frac{B}{2\mu}\bar{s}_{22}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{46}$$

$$\dot{\alpha}_{33} = \dot{e}\left[B\bar{t}_{12}\bar{s}_{33} - \frac{B}{2\mu}\bar{s}_{33}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{47}$$

$$\dot{\alpha}_{12} = \dot{e}\left[B\bar{t}_{12}^2 - \tfrac{1}{2}c_2\alpha_{11} - \tfrac{1}{2}c_1\alpha_{22} - \frac{B}{2\mu}\bar{t}_{12}^2(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{48}$$

$$\dot{k} = \dot{e}\left[C\bar{t}_{12} - \frac{C}{2\mu}\bar{t}_{12}(\bar{s}_{11} + \bar{t}_{22})(c_1 + c_2)\right] \tag{49}$$

where

$$A = \frac{3\mu^2}{k^2(3\mu + H')}, \quad B = (1-\beta)\frac{2H'}{k^2(3\mu + H')}, \quad C = \beta\frac{\mu H'}{k(3\mu + H')}.$$

Remark 4. Apparently if $c_1 + c_2 = 0$—which holds for stress rates with skew tensor $\mathbf{A}$—then the last terms in eqns (41)–(49) vanish.

The differential equation system, eqns (41)–(49), is solved numerically, using the classic fourth-order Runge–Kutta method in the next paragraph. Solutions for the elastic–perfectly plastic case ($H' = 0$) lead to simpler differential equations.

Table 5 shows the equations obtained by Moss (1984) for the Jaumann, Truesdell, and Green–Naghdi rates. Table 5 also contains the equations for the Sowerby–Chu and Durban–Baruch derivatives that have not been published (to the author's knowledge).

In Table 5, auxiliary variable θ is defined by

$$s_{11} = k_0 \sin\theta \quad \text{and} \quad s_{12} = k_0 \cos\theta$$

where k_0 is the yield point associated with pure shear.

Numerical results

Numerical calculations were made first for the elastic case. Figures 2 and 3 show the change of dimensionless stresses t_{12}/μ and t_{11}/μ, calculated on the basis of relationships given in Table 3, as a function of $e/2$.

Results for the elastic–perfectly plastic case are illustrated in Figs 4 and 5, the curves resulting from the numerical solutions of differential equations given in Table 5. For the sake of compatibility with the results of Moss (1984), the dimensionless values of deviatoric stress components are illustrated. A value $k_0/\mu = 0.0577$, used also by Moss (1984), has been used in the calculations.

Values of stresses obtained for elastic–isotropic hardening are illustrated in Figs 6 and 7 ($\beta = 1$), while the change of stress component t_{12} in the neighborhood of elastic–plastic transition is shown magnified in Fig. 8.

Figures 9–11 apply to elastic–plastic kinematic hardening ($\beta = 0$).

Table 5

Truesdell	$\dfrac{d\theta}{de} = \dfrac{2}{\sqrt{3}}\cos^2\theta - \sin\theta\left(\dfrac{t_{22}}{k_0} + \dfrac{\mu}{k_0}\right)$ $\dfrac{dt_{22}}{de} = k_0 \sin\theta\cos\theta\left[\dfrac{2}{3}\sin\theta + \dfrac{1}{\sqrt{3}}\left(\dfrac{t_{22}}{k_0} + \dfrac{\mu}{k_0}\right)\right]$
Jaumann	$\dfrac{d\theta}{de} = 1 - \dfrac{\mu}{k_0}\sin\theta$
Green–Naghdi	$\dfrac{d\theta}{de} = \dfrac{4}{e^2+4} - \dfrac{\mu}{k_0}\sin\theta$
Sowerby–Chu	$\dfrac{d\theta}{de} = \dfrac{2}{e^2+4} - \dfrac{\mu}{k_0}\sin\theta$
Durban–Baruch	$\dfrac{d\theta}{de} = \dfrac{13}{2\sqrt{(39)}} + \dfrac{5}{2\sqrt{(39)}}\sin^2\theta - \dfrac{\sin\theta}{2}\left(\dfrac{2\mu}{k_0} + \dfrac{t_{11}}{k_0}\right)$ $\dfrac{dt_{11}}{de} = k_0\cos\theta\left[\dfrac{3}{2} + \dfrac{35}{78}\sin^2\theta - \dfrac{7}{2\sqrt{(39)}}\sin\theta\left(\dfrac{2\mu}{k_0} + \dfrac{t_{11}}{k_0}\right)\right]$

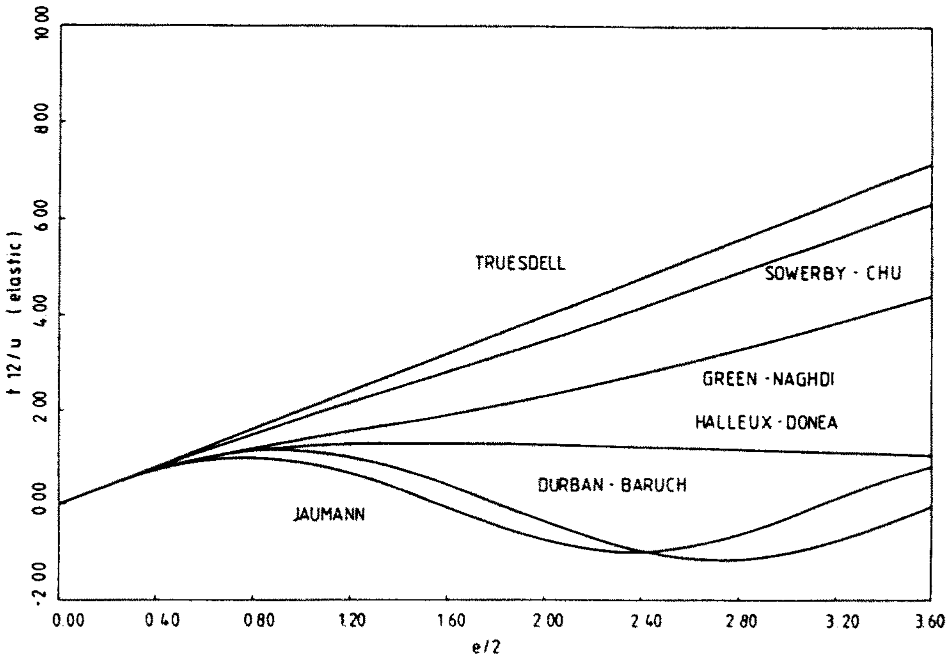

Fig. 2. Shear stress vs shear strain: elastic material.

Solutions, illustrated in Figs 12 and 13, are obtained for combined isotropic–kinematic hardening ($\beta = 0.5$), using material properties $\mu = 80000$ MPa, $H'/\mu = 0.1$, $k_0/\mu = 0.0577$ for the hardening models.

Discussion of numerical results

As Figs 2 and 3 illustrate, the Durban–Baruch derivative displays oscillating properties similar to the Jaumann derivative. A comparison of the Sowerby–Chu and Green–Naghdi

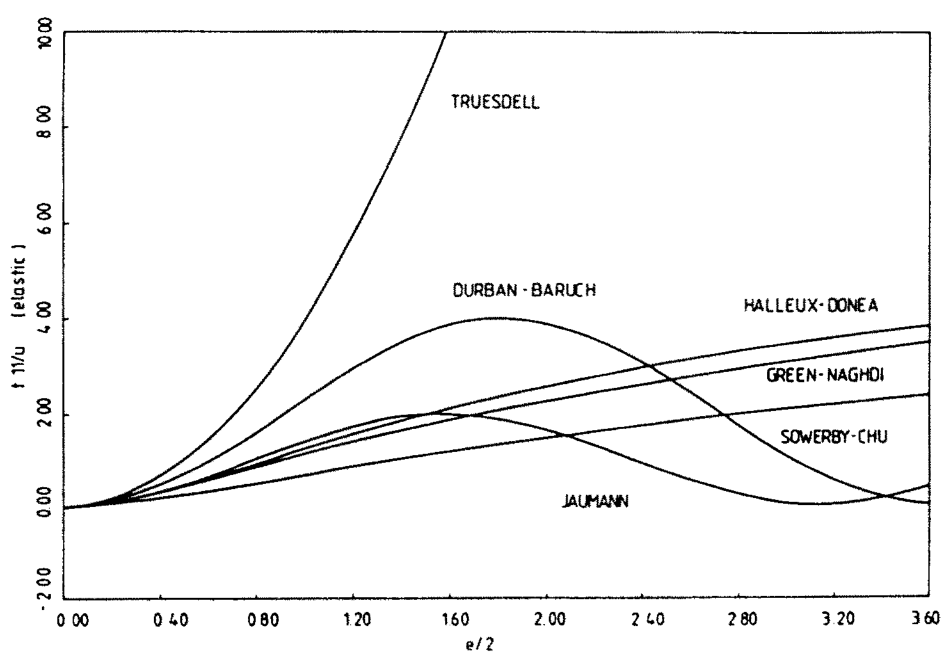

Fig. 3. Normal stress vs shear strain: elastic material.

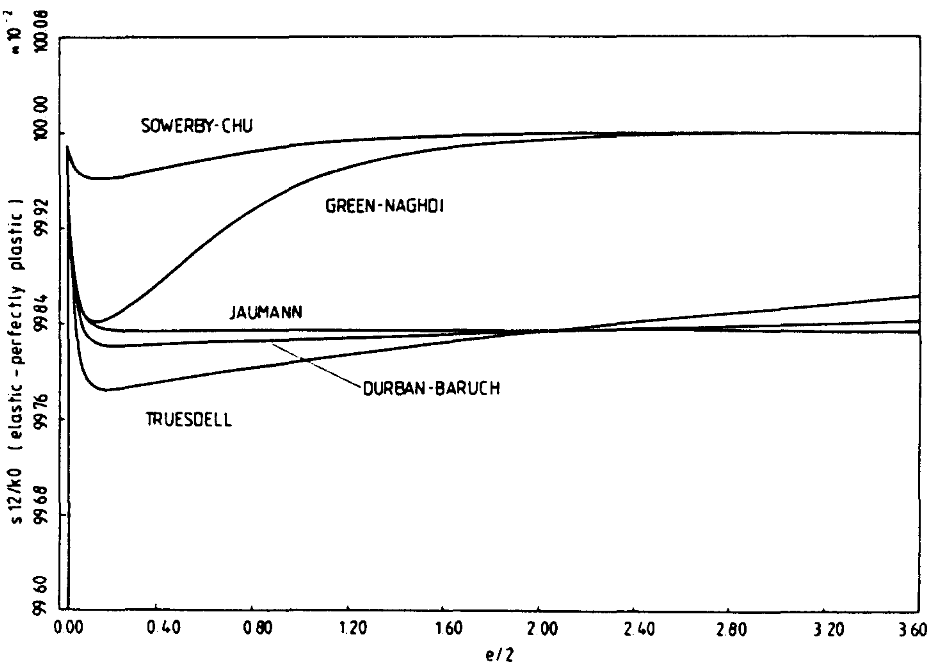

Fig. 4. Shear stress vs shear strain: elastic–perfectly plastic material.

derivatives shows that the Sowerby–Chu rate results in a smaller normal stress therefore the curve of stress component t_{12} which is closer to the linear characteristic than the Green–Naghdi rate.

With the calculations reproduced, the solutions for the elastic–perfectly plastic case (Figs 4 and 5) contain also the results of Moss (1984). Moss was the first to find for the Truesdell, Green–Naghdi, and Jaumann stress rates that instability occurred in stress components t_{12} in the elastic–plastic transition for the elastic–perfectly plastic case.

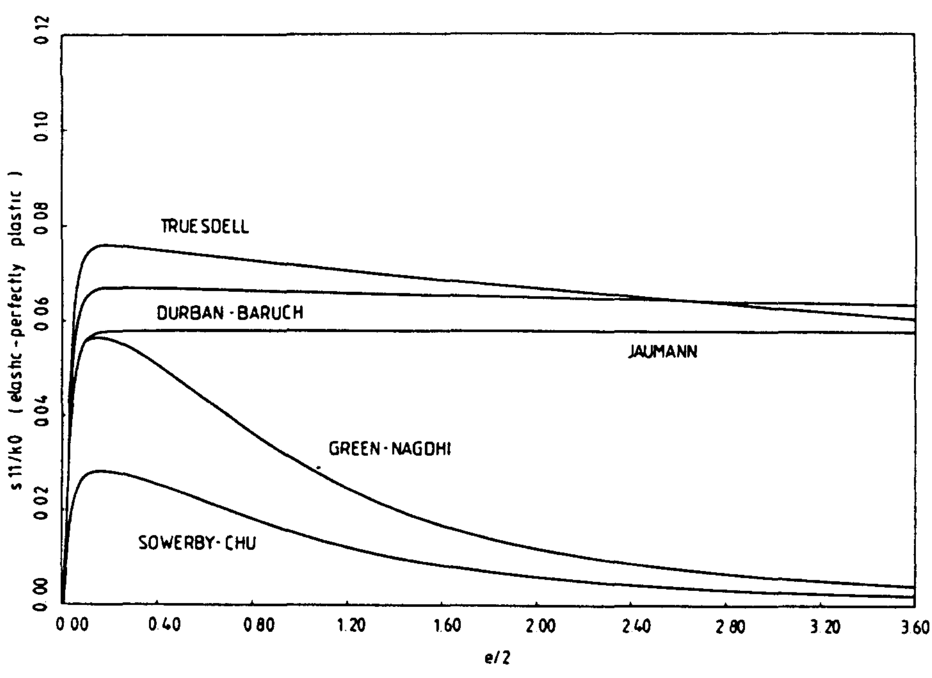

Fig. 5. Normal stress vs shear strain: elastic–perfectly plastic material.

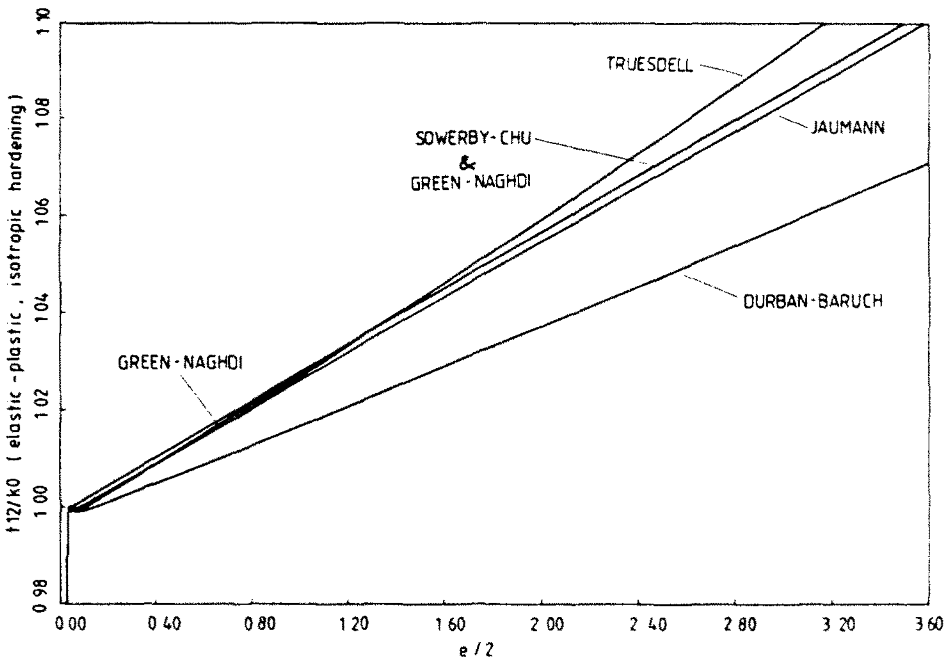

Fig. 6. Shear stress vs shear strain: elastic–plastic isotropic hardening material.

The Durban–Baruch derivative shows also instability. Least instability is found for the Sowerby–Chu rate from among the stress rates investigated.

The shear stress is approximately linear for all five derivatives in the elastic–plastic, isotropic hardening case (Fig. 6). However, a magnified representation of the section of elastic–plastic transition (Fig. 8) shows slight instability for the derivatives except for the Sowerby–Chu rate. It can be seen in Fig. 7, illustrating the change of stress t_{11}, that the curves for the Durban–Baruch and Truesdell stress rates are very steep.

As has been found by different authors (Nagtegaal and de Jong, 1982; Key, 1984;

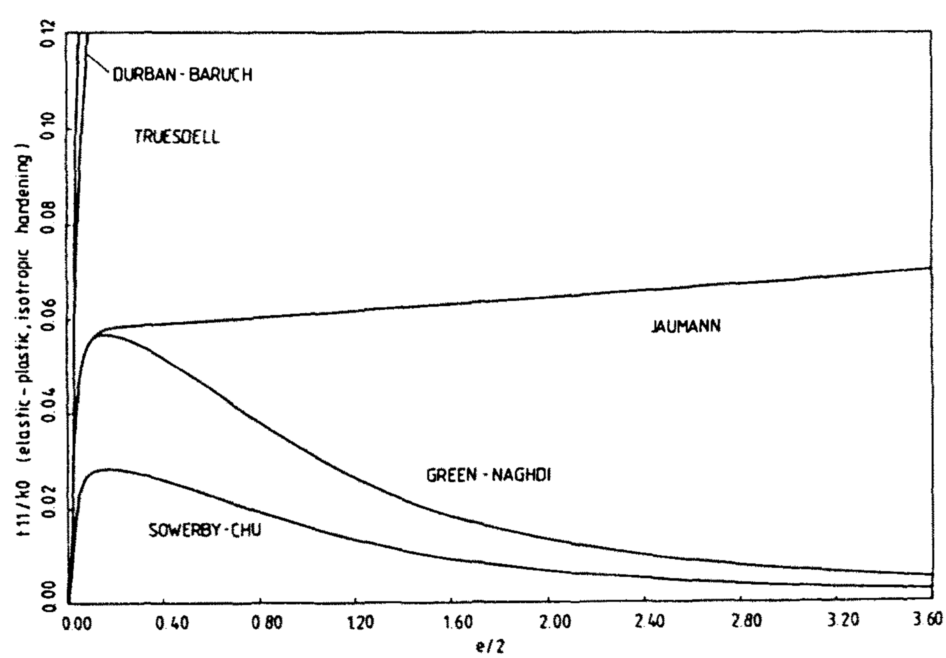

Fig. 7. Normal stress vs shear strain: elastic–plastic isotropic hadening material.

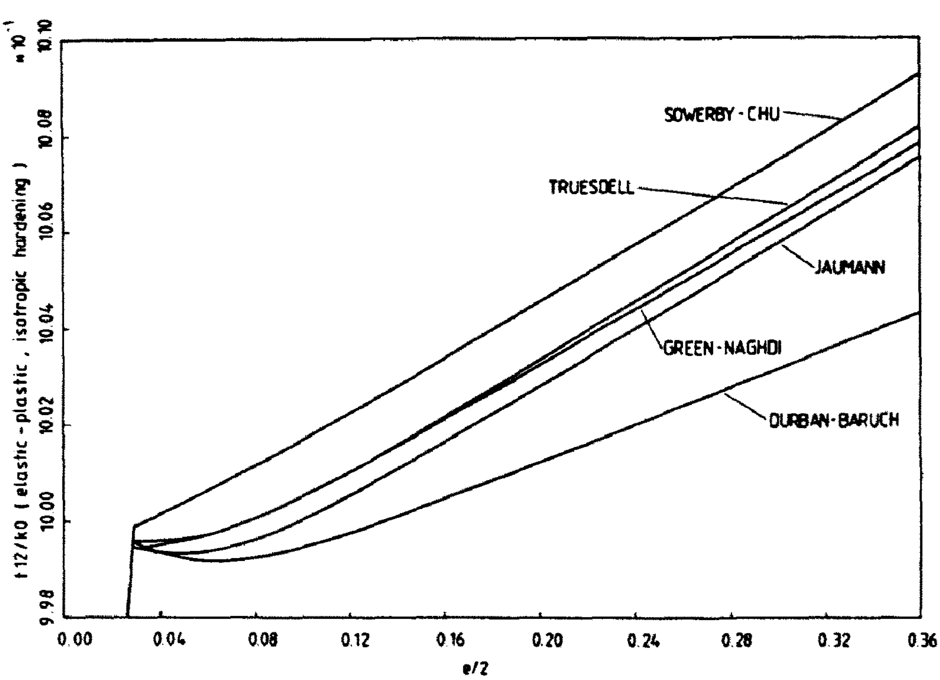

Fig. 8. Shear stress vs shear strain: elastic–plastic isotropic hardening material (a magnified representation of the section of the elastic–plastic transition).

Johnson and Bammann, 1984; Atluri, 1984b; Dafalias, 1983; Lee *et al.*, 1983) the solutions show an oscillating behaviour for the Jaumann derivative in the case of elastic–plastic, kinematic hardening. As compared with the Green–Naghdi derivative, the Sowerby–Chu derivative results in a solution closer to linear in stress t_{12}. With the section of elastic–plastic transition in Fig. 9 magnified, a slight instability can be experienced in the solutions also in this case, except for the Sowerby–Chu rate. This phenomenon has not been detected by calculations made on the basis of a rigid-plastic model. Results differing from the solutions illustrated in the figures are obtained also in approximate calculations where the total strain

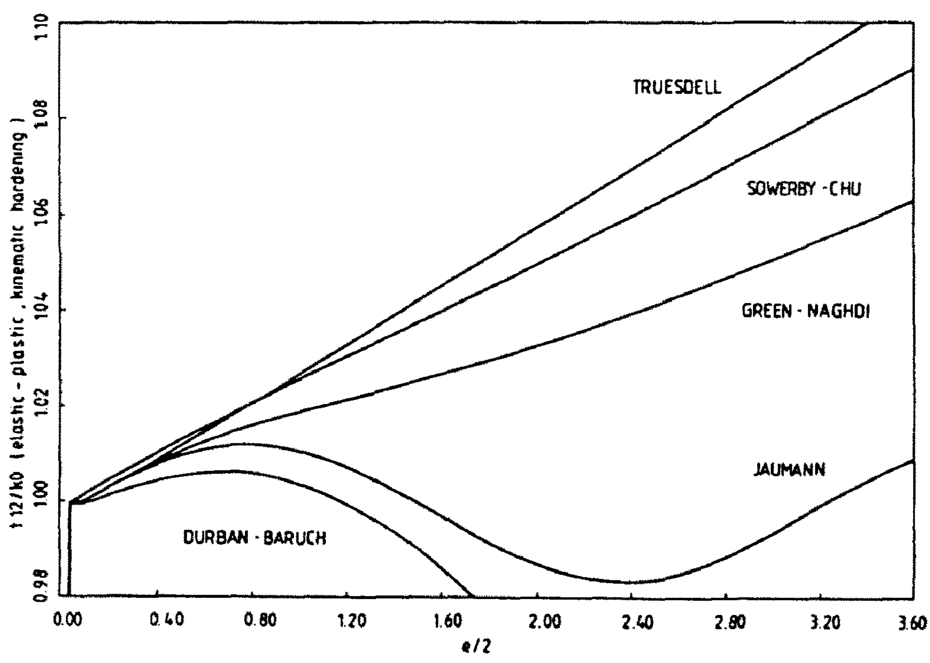

Fig. 9. Shear stress vs shear strain: elastic–plastic kinematic hardening material.

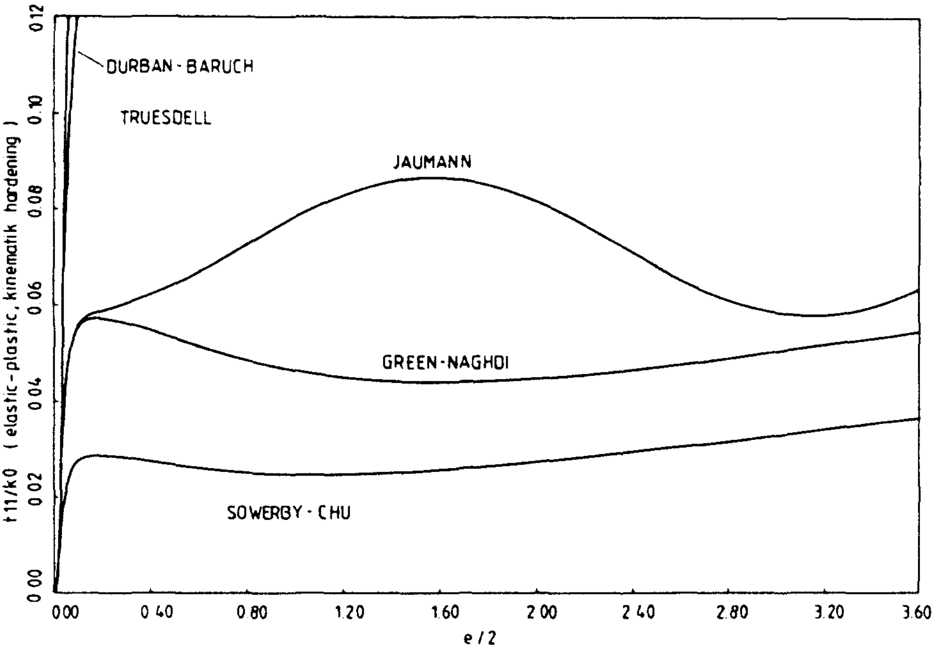

Fig. 10. Normal stress vs shear strain: elastic–plastic kinematic hardening material.

rate is used instead of the plastic one to produce the rate of the "back-stress" tensor (Atluri, 1984b). Very steep curves are obtained for the Durban–Baruch and Truesdell derivatives in normal stresses also in this model, similar to isotropic hardening.

Results obtained for combined hardening appear as the average of solutions associated with isotropic and kinematic hardening when $\beta = 0.5$ as shown also by a comparison of Figs 6, 9, 12, and 7, 10, 13.

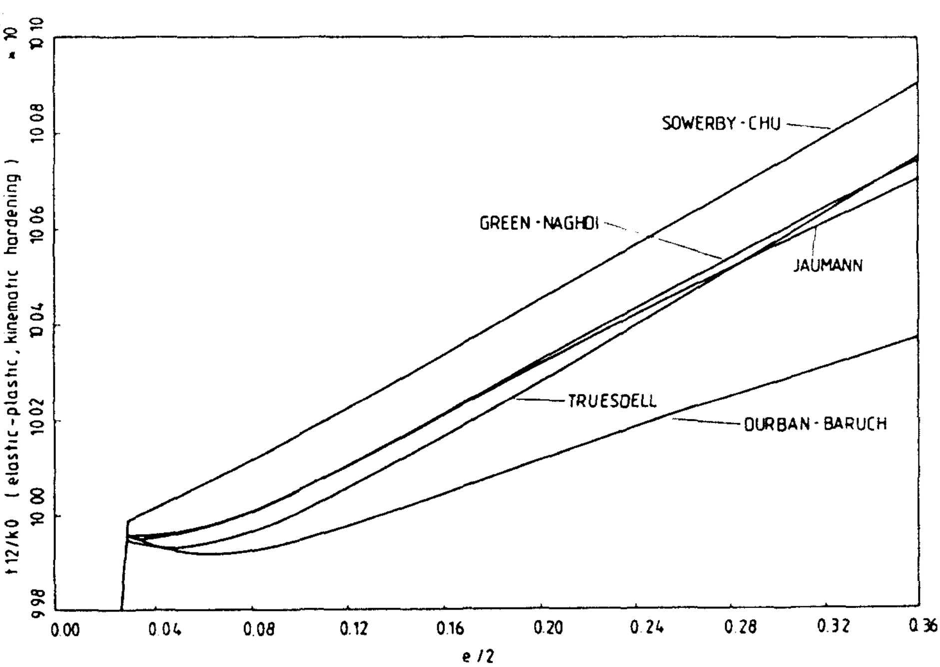

Fig. 11. Shear stress vs shear strain: elastic–plastic kinematic hardening material (a magnified representation of the section of the elastic–plastic transition).

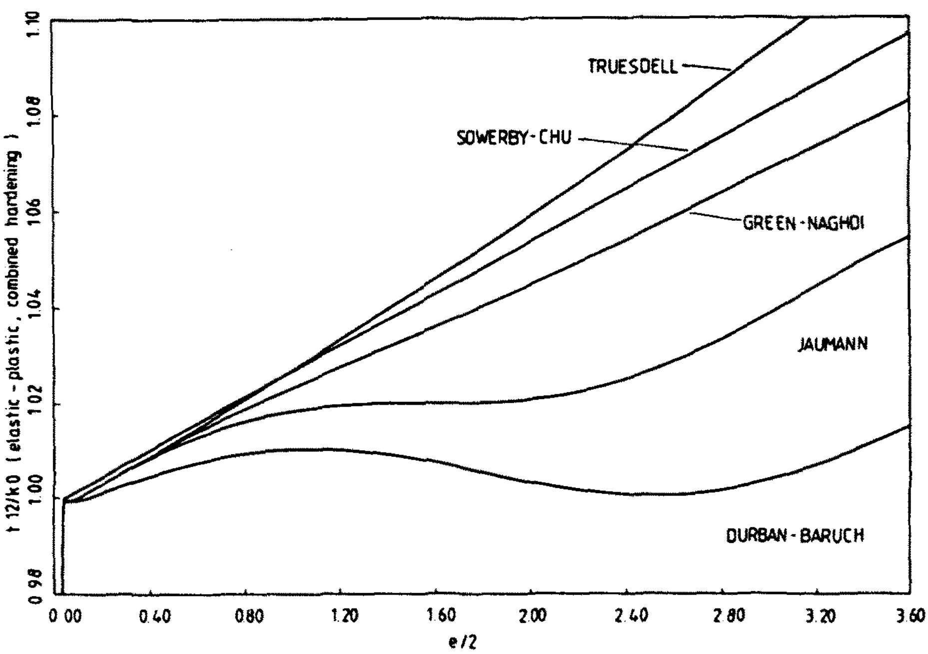

Fig. 12. Shear stress vs shear strain: elastic–plastic combined isotropic–kinematic hardening material.

CONSTITUTIVE RELATIONS

The simple elastic shear was analysed by eqn (37). Many authors showed that the Jaumann stress rate results in residual stresses for a closed strain path (Kojic and Bathe, 1987; Kleiber, 1986). Lately the application of the Green–Naghdi stress rate has been preferred instead of that of Jaumann (Flanagan and Taylor, 1987; Hughes, 1984; Johnson and Bammann, 1984; Kim and Oden, 1985). But Simo and Pister (1984) pointed out

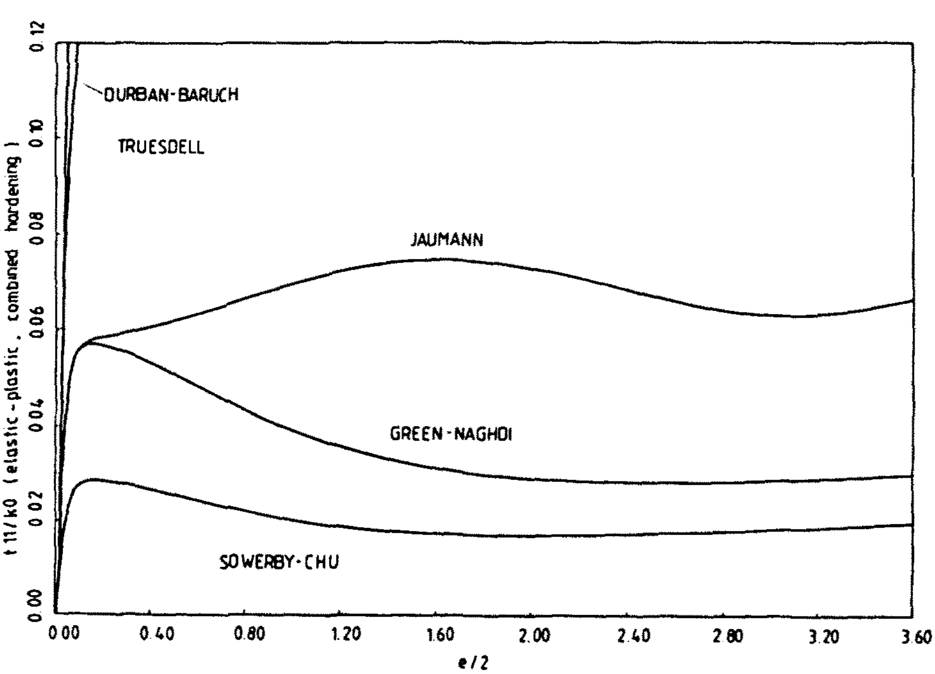

Fig. 13. Normal stress vs shear strain: elastic–plastic combined isotropic–kinematic hardening material.

that neither the use of Jaumann nor Green–Naghdi stress rates in eqn (37) gives elastic characteristics.

It was apparent from the comparison of stress rates that the Sowerby–Chu rate gives acceptable results (there are no oscillation and instability at the elastic–plastic limit).

For simple elastic shear the following solution was given (Halleux and Donea, 1985):

$$t_{11} = -t_{22} = 2\mu \sin(\beta) \ln\left(\frac{\sin(\beta)+1}{\cos(\beta)}\right)$$

$$t_{12} = 2\mu \cos(\beta) \ln\left(\frac{\sin(\beta)+1}{\cos(\beta)}\right). \tag{50}$$

This solution is given on Figs 2 and 3 for comparison with others. It is also included in the solution for the Sowerby–Chu stress rate (see row 4 on Table 3). The only differences are the last two terms which are multiplied by μ. This fact indicates that the right-hand side of eqn (37) must be modified in order to obtain solution (50).

The Doyle–Ericksen formula for hyperelastic materials is (see p. 204 of Marsden and Hughes (1983))

$$\tau = 2\rho_0 \frac{\partial \psi}{\partial \mathbf{g}} \tag{51}$$

where ψ is the free energy, ρ_0 the density in reference configuration C_0, and $\mathbf{g}$ the spatial metric tensor.

Proposition 2. Denoting the symmetric part of $\mathbf{L}_E$ by

$$\mathbf{D}_E = (\mathbf{L}_E)_S \tag{52}$$

and using the new derivation rule on both sides of the (spatial) Doyle–Ericksen formula (51) we obtain

$$\hat{\tau}_E = 4\rho_0 \frac{\partial^2 \psi}{\partial \mathbf{g}\, \partial \mathbf{g}} : \mathbf{D}_E. \tag{53}$$

Proof. If the new derivation rule is applied to the Doyle–Ericksen formula, then

$$\hat{\tau}_E = 2 \frac{\partial^2 \psi}{\partial \mathbf{g}\, \partial \mathbf{g}} : \hat{\mathbf{g}}_E. \tag{54}$$

As $\mathbf{g}$ is given in covariant coordinates (g_{ab})

$$\hat{\mathbf{g}}_E = \dot{\mathbf{g}} + \mathbf{L}_E^T \mathbf{g} + \mathbf{g}\mathbf{L}_E. \tag{55}$$

Since $\dot{\mathbf{g}} = 0$, therefore

$$\hat{\mathbf{g}}_E = \mathbf{L}_E^T \mathbf{g} + \mathbf{g}\mathbf{L}_E = 2(\mathbf{L}_E)_S = 2\mathbf{D}_E. \tag{56}$$

By substituting eqn (56) into eqn (54) we obtain eqn (53). □

Equation (53) is the new rate form of the Doyle–Ericksen formula.

The use of logarithmic strain has appeared recently in the constitutive equations of elastic and elastic–plastic materials (Bathe *et al.*, 1985; Halleux and Donea, 1985).

Proposition 3. The relation between $\mathbf{D}_E$ and logarithmic strain $\ln \mathbf{V}$ is

$$\mathbf{D}_E = \overset{*}{\overline{\ln \mathbf{V}}} \tag{57}$$

where $\overset{*}{\overline{\ln \mathbf{V}}}$ is the Sowerby–Chu rate of $\ln \mathbf{V}$ relative to the Eulerian triad

$$\overset{*}{\overline{\ln \mathbf{V}}} = \overset{\cdot}{\overline{\ln \mathbf{V}}} - \boldsymbol{\Omega}_{\mathrm{E}} \ln \mathbf{V} + \ln \mathbf{V}\, \boldsymbol{\Omega}_{\mathrm{E}}.$$

Proof. Chu (1986) gives the expression for $\mathbf{L}$ as (see eqn (37) of Chu (1986))

$$\mathbf{L} = \overset{*}{\overline{\ln \mathbf{V}}} + \boldsymbol{\Omega}_{\mathrm{E}} - \mathbf{F}\boldsymbol{\Omega}_{\mathrm{L}}\mathbf{F}^{-1}. \tag{58}$$

Substituting eqn (58) into eqn (25) we obtain

$$\mathbf{L}_{\mathrm{E}} = \overset{*}{\overline{\ln \mathbf{V}}} + \boldsymbol{\Omega}_{\mathrm{E}} \tag{59}$$

from which, as eqn (27) holds, thus

$$(\mathbf{L}_{\mathrm{E}})_{\mathrm{S}} = \overset{*}{\overline{\ln \mathbf{V}}} = \mathbf{D}_{\mathrm{E}}. \qquad \square \tag{60}$$

Equation (53) is analogous with the Lie derivative of eqn (51) given by Simo and Marsden (1984) and Marsden and Hughes (1983)

$$\check{\boldsymbol{\tau}} = 4\rho_0 \frac{\partial^2 \psi}{\partial \mathbf{g}\, \partial \mathbf{g}} : \mathbf{D}. \tag{61}$$

As the stress $\check{\boldsymbol{\tau}}$ is conjugate to the strain rate $\mathbf{D}$ in eqn (61), $\check{\boldsymbol{\tau}}_{\mathrm{E}}$ is conjugate to the logarithmic strain rate $\mathbf{D}_{\mathrm{E}}$ in eqn (53). If these conjugate pairs are used in eqn (37), respectively, the same solution for simple shear is obtained (first row of Table 4). On the other side eqn (61) includes the Jaumann rate analogous with

$$\overset{\circ}{\boldsymbol{\tau}} = \mathbf{C}(\mathbf{g}, \boldsymbol{\tau}) : \mathbf{D}_{\mathrm{E}} \tag{62}$$

where $\mathbf{C}$ is the constitutive tensor. It follows from eqn (62) for simple hypoelastic materials that

$$\overset{\circ}{\boldsymbol{\tau}} = 2\mu \mathbf{D}_{\mathrm{E}} + \lambda \mathbf{I}\ \mathrm{tr}\ \mathbf{D}_{\mathrm{E}} \tag{63}$$

which gives solution (50) for simple shear. Many authors (Hoger, 1987; Atluri, 1984a) showed that $\boldsymbol{\tau}$ and $\ln \mathbf{V}$ are conjugate stress and strain measures in the isotropic elastic case. Atluri (1985) gave the simple hyperelastic relation between $\boldsymbol{\tau}$ and $\ln \mathbf{V}$ in the form of

$$\boldsymbol{\tau} = 2\mu \ln \mathbf{V} + \lambda \mathbf{I}\ \mathrm{tr}\ (\ln \mathbf{V}). \tag{64}$$

Using the Sowerby–Chu derivation on both sides of eqn (64), eqn (63) is obtained. Equation (63) is the physically objective rate form of eqn (64) on configuration C_t.

The solution to the differential equation for simple shear

$$\frac{\mathrm{d}^2 t_{12}}{\mathrm{d}\beta^2} + t_{12} = -2\mu \tan(\beta) \tag{65}$$

received from eqn (63), is also identical to eqn (50). However, eqns (50) can be obtained directly from eqn (64).

Equation (63) on configuration $C_{\lambda t}$ is

$$\overset{\cdot}{\overline{\mathbf{R}_{\mathrm{E}}^{\mathrm{T}} \boldsymbol{\tau} \mathbf{R}_{\mathrm{E}}}} = 2\mu\, \overset{\cdot}{\overline{\ln \mathbf{V}}} + \lambda \mathbf{I}\ \mathrm{tr}\ (\overset{\cdot}{\overline{\ln \mathbf{V}}}) \tag{66}$$

or in another form

$$\overline{\mathbf{R}_E^T \tau \mathbf{R}_E}^{\cdot} = 2\mu\dot{\lambda}\lambda^{-1} + \lambda \mathbf{I}\ \mathrm{tr}\ (\dot{\lambda}\lambda^{-1}). \tag{67}$$

Halleux and Donea (1985) solved eqn (67) in a geometric way and then transformed the solution on the instantaneous (C_t) configuration. This procedure leads to eqns (50).

Finally we note that among the stress rates considered in Table 4, only the Truesdell (first row) and solution (50) satisfy the local universal relation obtained by Wineman and Gandhi (1984) for isotropic, elastic simple shear

$$t_{11} - t_{22} = e t_{12}. \tag{68}$$

CONCLUSIONS

A new derivation rule was introduced, by means of which two objective stress rates (eqns (17) and (18)) were derived.

As the Jaumann rate is equal to the average of the Oldroyd and Cotter–Rivlin rates, likewise the Sowerby–Chu rate is the average of the two new stress rates (20).

The new derivation rule was applied to the hyperelastic Doyle–Ericksen formula, resulting in a new rate form. In this rate form, the new stress rate (17) is conjugate to the Sowerby–Chu derivative of the logarithmic left stretch tensor, in the current state.

Furthermore, the paper compares some stress rates for simple shear. The comparative investigations showed similarity between the Durban–Baruch and Jaumann stress rates on the one hand, and between the Sowerby–Chu and Green–Naghdi stress rates on the other. It was shown that the different rates result in different behaviour. Usually it is questionable which rate can be used in a constitutive equation. It was shown that for simple shear the Sowerby–Chu rate gives acceptable results.

Starting from this observation a new constitutive equation for hypoelastic materials was given which includes the Sowerby–Chu stress rate. The new equation leads to the known solution for simple shear.

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